GEOMETRY MIDTERM EXAM (FEB 2026)

Answer all the questions. Total 30 marks.

(You may use any theorem proved in class, but you must state it clearly.)

Note. All affine spaces below are assumed to be finite-dimensional.

(1) [6 points] Let φ be an orientation-preserving isometry (rigid motion) of Euclidean 3-space, $\mathbb{R}^3$. Prove that φ can be uniquely decomposed as the composition of a rotation about an axis ℓ and a translation by a vector v parallel to ℓ:

$$\varphi = t_v \circ R_{\ell,\theta},$$

where θ is the rotation angle.

(2) [6 points] Let $\varphi : \mathcal{A} \to \mathcal{A}$ be an affine transformation such that $\varphi^2 = \mathrm{Id}$ (an involution), where $\mathcal{A}$ is an affine space over a field of characteristic $\neq 2$.

- (a) Prove that φ must have a non-empty set of fixed points.
- (b) Classify these involutions based on the dimension of their fixed-point set (e.g., reflection in a point, reflection in a hyperplane).

(3) [6 points] Let E be an euclidean vector space and $C \subset E$ be a convex set. We define

$$\check{C} := \{u \in E : \langle u, v\rangle \leq 1 \ \text{ for all } v \in C\}.$$

- (a) Prove that $\check{C}$ is convex.
- (c) Let $P \subset E$ be a convex polyhedron, the interior of which contains 0. Prove that $\check{P}$ is a convex polyhedron.

(4) [4 points] Let U be a non-empty open subset on the unit sphere S^2. Prove that there exists no mapping $f : U \longrightarrow \mathbb{R}^2$ that preserves the distances (i.e., $d(x, y) = \|f(x) - f(y)\|$, $\forall x, y \in S^2$, where d is the induced distance on S^2).

(5) [5 points] Let $\mathcal{F} = \{K_1, K_2, \ldots, K_m\}$ be a finite family of convex subsets of $\mathbb{R}^n$, with $m > n$. Prove that if the intersection of every subfamily of size $n + 1$ is non-empty, then the intersection of the entire family is non-empty:

$$\bigcap_{i=1}^{m} K_i \neq \emptyset.$$

(6) [3 points] Regular pentagons and regular hexagons with the same side length are glued together so that there are three polygons at each vertex, and the result is a convex polyhedron. How many pentagons are needed? (Hint: use the Euler formula)